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# Certain family of analytic and p-valent functions associated with subordination

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**Abstract:**
The main aim of this paper is to introduce a new class $\mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$ of $p-$valent and analytic function $f(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ and obtained the results of coefficient estimates, growth and distortion theorem, radii of close to convexity, starlikeness and convexity, colousure theorem, weighted mean, arithmetic mean, linear combination for the class.

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## 1. Introduction, Definitions and Preliminaries

Let $\mathrm{A}_p$ denote the class of all functions $f(z)$ of the form

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad \left(p\in\mathbb{N}=\{1,2,3,\ldots\}\right) \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk $\mathrm{U}=\{z:z\in\mathbb{C}:|z|<1\}$. Note that $\mathrm{A}_1:=\mathrm{A}$ the class of analytic functions and let $\mathrm{S}$, the subclass of $\mathrm{A}$ consisting of all univalent functions $f$ in $\mathrm{U}$. A function $f\in\mathrm{A}$ is said to be starlike of order $\alpha(0\le\alpha<1)$ in $\mathrm{U}$ if it satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha.$$

This class is denoted by $\mathrm{S}^{*}(\alpha)$ and note that $\mathrm{S}^{*}(0)=\mathrm{S}^{*}$. The class $\mathrm{S}^{*}(\alpha)$ was introduced by Robertson. It is well-known that $\mathrm{S}^{*}(\alpha)\subset\mathrm{S}^{*}\subset\mathrm{S}$.

Let $P(z)$ and $Q(z)$ be analytic in $\mathrm{U}$. Then the function $P(z)$ is said to subordinate to $Q(z)$ in $\mathrm{U}$ written by

$$P(z)\prec Q(z) \qquad (z\in\mathrm{U}), \tag{1.2}$$

if there exists a function $w(z)$ which is analytic in $\mathrm{U}$ with $w(0)=0$ and $|w(z)|<1$ $(z\in\mathrm{U})$, and such that $P(z)=Q(w(z))(z\in\mathrm{U})$. From the definition of the subordinations, it is easy to show that the subordination (1.1) implies that

$$P(0)=Q(0) \qquad and \qquad P(\mathrm{U})\subset Q(\mathrm{U}). \tag{1.3}$$

In particular, if $Q(z)$ is univalent in $\mathrm{U}$, then the subordination (1.1) is equivalent to the condition (1.2).

Introduced by the classes $\mathrm{S}^{*}(\alpha)$ and $\mathrm{M}(\beta)$, we define a new class for certain $p$ - valent functions.

**Definition 1.1.** Let $\alpha$ be the real number such that $0\le\alpha<1$. The function $f\in\mathrm{A}_p$ belongs to the class $\mathrm{S}^{p}(\alpha)$ if $f$ satisfies the following inequality

$$Re\left(\frac{zf'(z)}{pf(z)}\right)>\alpha \qquad (z\in\mathrm{U},p\in\mathbb{N}). \tag{1.4}$$

**Definition 1.2.** Let $\alpha$ be then real number such that $0\le\alpha<1$. The function $f\in\mathrm{A}_p$ belongs to the class $\mathrm{C}^{p}(\alpha)$ if $f$ satisfies the following inequality

$$Re\left\{\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha \qquad (z\in\mathrm{U}). \tag{1.5}$$

**Definition 1.3.** Let $\alpha$ be the real number such that $0\le\alpha<1$. The function $f\in\mathrm{A}_p$ belongs to the class $\mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$ if $f$ satisfies the following

$$\left\{\frac{\dfrac{zf'(z)}{pf(z)}-1}{(p-\alpha)\mu+\alpha\lambda-\lambda\dfrac{zf'(z)}{pf(z)}}\right\}\prec\frac{1+Az}{1+Bz} \tag{1.6}$$

for $0 \le \alpha < 1,\ 0 \le \lambda < \mu \le 1,\ -1 \le B < A \le 1$

**2. Coefficient Estimates of Basic Properties for** $\mathrm{M}^p(\alpha,\mu,\lambda,A,B)$

**Theorem 2.1.** The function $f(z)$ defined by (1.1) is in the class $\mathrm{M}^p(\alpha,\mu,\lambda,A,B)$ if and only if

$$\sum_{n=1}^{\infty}\Big[n-\big[p\big((p-\alpha)\mu+\alpha\lambda\big)+(n+p)\lambda\big](A+1)+B(n+2p)\Big]a_{n+p} \le p(A+1)\big[(p-\alpha)\mu+\lambda(\alpha-1)\big] \tag{2.1}$$

Proof: Since $f(z)\in \mathrm{M}^p(\alpha,\mu,\lambda,A,B)$, it suffices to show that

$$F(z)=\left[\frac{\frac{zf'(z)}{pf(z)}-1}{(p-\alpha)\mu+\alpha\lambda-\lambda\frac{zf'(z)}{pf(z)}}\right]\prec\frac{1+Az}{1+Bz}$$

$$F(z)=\frac{1+Aw(z)}{1+Bw(z)}$$

$$F(z)[1+Bw(z)]=1+Aw(z)$$

$$w(z)[BF(z)-A]=1-F(z)$$

$$w(z)=\frac{F(z)-1}{A-BF(z)}$$

$$|w(z)|<1$$

$$\left|\frac{F(z)-1}{A-BF(z)}\right|<1$$

$$\left|\frac{\left[\frac{\frac{zf'(z)}{pf(z)}-1}{(p-\alpha)\mu+\alpha\lambda-\lambda\frac{zf'(z)}{pf(z)}}\right]-1}{A-B\left[\frac{\frac{zf'(z)}{pf(z)}-1}{(p-\alpha)\mu+\alpha\lambda-\lambda\frac{zf'(z)}{pf(z)}}\right]}\right|<1$$

$$\left|\frac{zf'(z)-pf(z)-pf(z)[(p-\alpha)\mu+\alpha\lambda]+\lambda zf'(z)}{Apf(z)[(p-\alpha)\mu+\alpha\lambda]-A\lambda zf'(z)-Bzf'(z)+Bpf(z)}\right|<1$$

Now,

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$$

$$zf'(z)=pz^p+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}$$

$$zf'(z)-pf(z)-pf(z)[(p-\alpha)\mu+\alpha\lambda]+\lambda zf'(z)$$

$$=pz^p+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}-p\left(z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\right)$$

$$-p\left(z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\right)\big[(p-\alpha)\mu+\alpha\lambda\big]+\lambda\left(pz^p+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}\right)$$

$$=\sum_{n=1}^{\infty}\big[n-p((p-\alpha)\mu+\alpha\lambda)+(n+p)\lambda\big]a_{n+p}z^{n+p}-\big\{p[(p-\alpha)\mu+\alpha\lambda]-p\lambda\big\}z^p$$

and

$$Apf(z)[(p-\alpha)\mu+\alpha\lambda]-A\lambda zf'(z)-Bzf'(z)+Bpf(z)$$

$$=Ap\left(z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\right)\big[(p-\alpha)\mu+\alpha\lambda\big]-Apz^p-A\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}$$

$$-Bp\left(pz^p+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}\right)+Bp\left(z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\right)$$

$$=\sum_{n=1}^{\infty}\left[A\left[p((p-\alpha)\mu+\alpha\lambda)+(n+p)\lambda\right]-B(n+2p)\right]a_{n+p}z^{n+p}$$
$$+A\left[p((p-\alpha)\mu+\alpha\lambda)-p\lambda\right]z^{p}$$

$$\left|\frac{\sum_{n=1}^{\infty}\left[n-p((p-\alpha)\mu+\alpha\lambda)+(n+p)\lambda\right]a_{n+p}z^{n+p}-\left[p[(p-\alpha)\mu+\alpha\lambda]-p\lambda\right]z^{p}}{\sum_{n=1}^{\infty}\left[A\left[p((p-\alpha)\mu+\alpha\lambda)+(n+p)\lambda\right]-B(n+2p)\right]a_{n+p}z^{n+p}+A\left[p((p-\alpha)\mu+\alpha\lambda)-p\lambda\right]z^{p}}\right|<1.$$

Since $Re(z)<|z|$. After choosing the values of $z$ on real axis and letting $z\to 1$ we get

$$\sum_{n=1}^{\infty}\left[n-p((p-\alpha)\mu+\alpha\lambda)+(n+p)\lambda\right]a_{n+p}-\left\{p[(p-\alpha)\mu+\alpha\lambda]-p\lambda\right\}$$
$$\leq\sum_{n=1}^{\infty}\left[A\left[p((p-\alpha)\mu+\alpha\lambda)+(n+p)\lambda\right]-B(n+2p)\right]a_{n+p}$$
$$+A\left[p((p-\alpha)\mu+\alpha\lambda)-p\lambda\right]$$

after simplifying, we get

$$\sum_{n=1}^{\infty}\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]a_{n+p}\leq p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right].$$

**Corollary 2.2.** The function $f(z)$ defined by (1.1) is in the class $M^{p}(\alpha,\mu,\lambda,A,B)$ then

$$\sum_{n=1}^{\infty}a_{n+p}\leq\frac{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]} \tag{2.2}$$

and the inequality holds for

$$f(z)=z^{p}+\frac{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]}z^{n+p}. \tag{2.3}$$

**Theorem 2.3.** The function $f(z)$ defined by (1.1) is in the class $M^{p}(\alpha,\mu,\lambda,A,B)$ then

$$|z^{p}|+|z^{p+1}|\frac{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}\leq|f(z)|$$
$$\leq|z^{p}|-|z^{p+1}|\frac{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}. \tag{2.4}$$

Proof: From theorem 2.1 we have

$$\sum_{n=1}^{\infty}a_{n+p}\leq\frac{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]}$$

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}$$

$$|f(z)|\geq|z^{p}|+|z^{1+p}|\frac{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]}$$

and

$$|f(z)|\leq|z^{p}|-|z^{1+p}|\frac{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]}$$

therefore

$$|z^{p}|+|z^{p+1}|\frac{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}\leq|f(z)|$$
$$\leq|z^{p}|-|z^{p+1}|\frac{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}. \tag{2.5}$$

**Theorem 2.4.** The function $f(z)$ defined by (1.1) is in the class $\mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$ then

$$|pz^{p-1}|+|z^{p}|\frac{(1+p)p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}\le|f'(z)| \\ \le|pz^{p-1}|-|z^{p}|\frac{(1+p)p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]} \tag{2.6}$$

Proof: From theorem 2.1 we have

$$\sum_{n=1}^{\infty}a_{n+p}\le\frac{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]}$$

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}$$

Differentiating and simplifying we obtain:

$$f'(z)=pz^{p-1}+\sum_{n=1}^{\infty}(n+p)a_{n+p}z^{n+p-1}$$

$$|f'(z)|\ge|pz^{p-1}|+|z^{p}|\sum_{n=1}^{\infty}|a_{n+p}|\ge|pz^{p-1}|+(1+p)|z^{p}|\frac{(1+p)p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]}$$

and

$$|f'(z)|\le|pz^{p-1}|-|z^{p}|\sum_{n=1}^{\infty}|a_{n+p}|\le|pz^{p-1}|+(1+p)|z^{p}|\frac{(1+p)p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]}$$

therefore

$$|pz^{p-1}|+|z^{p}|\frac{(1+p)p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}\le|f'(z)| \\ \le|pz^{p-1}|-|z^{p}|\frac{(1+p)p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]} \tag{2.7}$$

## 3. Radii of Close-to-Convexity, Starlikeness and Convexity

Now we provide the radii of $p$ - valently close-to-convexity, starlikeness and convexity for the class $\mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$.

**Theorem 3.1.** Let $f\in\mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$. Then $f$ is $p$ - valently close-to-convex of order $\eta$ $(0\le\eta<p)$ in $|z|<R_1$, where

$$R_1=\inf_{n\in\mathbb{N}}\left\{\left[\left[\frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}\left(\frac{p-\eta}{n+p}\right)\right]\right]\right\}^{\frac{1}{n}} \tag{3.1}$$

Proof: Given $f\in\mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$ and $f$ is close-to-convex of order $\eta$, $(0\le\eta<p)$ we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|<p-\eta. \tag{3.2}$$

For the left hand side of (3.2), we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|=\left|\frac{pz^{p-1}+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n-1}}{z^{p-1}}-p\right|\le\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{n}.$$

The last expression is less than $(p-\eta)$ if

$$\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{n}<p-\eta$$

which implies

$$\sum_{n=1}^{\infty}\left(\frac{p+n}{p-\eta}\right)a_{p+n}z^{n}<1.$$

Using the fact that, $f\in\mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$ if and only if

$$\sum_{n=1}^{\infty}\frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}a_{n+p}\leq 1,$$

we can say that (3.2) is true if

$$\left(\frac{p+n}{p-\eta}\right)|z|^{n}\leq\frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}$$

$$\Rightarrow |z|^{n}\leq\frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}\left(\frac{p-\eta}{p+n}\right).$$

The last inequality leads us immediately to the disc $|z|<R_1$, where $R_1$ is given by (3.1).

**Theorem 3.2.** Let $f\in M^{p}(\alpha,\mu,\lambda,A,B)$. Then $f$ is $p$- valently starlike of order $\eta$ $(0\leq\eta<p)$ in $|z|<R_2$, where

$$R_2=\inf_{n\in\mathbb{N}}\left\{\left[\frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}\left(\frac{p-\eta}{n-p+\eta}\right)\right]\right\}^{\frac{1}{n}} \tag{3.3}$$

Proof: Given $f\in M^{p}(\alpha,\mu,\lambda,A,B)$ and $f$ is starlike of order $\eta$, $(0\leq\eta<p)$ we have

$$\left|\frac{zf'(z)}{f(z)}-p\right|<p-\eta. \tag{3.4}$$

For the left hand side of (3.4), we have

$$\left|\frac{zf'(z)}{f(z)}-p\right|=\left|\frac{pz^{p}+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}}{z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}}-p\right|\leq\frac{\sum_{n=1}^{\infty}na_{p+n}z^{n+p}}{z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{n+p}}.$$

The last expression is less than $(p-\eta)$ if

$$\frac{\sum_{n=1}^{\infty}na_{p+n}z^{n+p}}{z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{n+p}}<p-\eta$$

which implies

$$\sum_{n=1}^{\infty}\left(\frac{n-p+\eta}{p-\eta}\right)a_{p+n}|z|^{n}<1.$$

Using the fact that, $f\in M^{p}(\alpha,\mu,\lambda,A,B)$ if and only if

$$\sum_{n=1}^{\infty}\frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}a_{n+p}\leq 1,$$

we can say that (3.4) is true if

$$\left(\frac{n-p+\eta}{p-\eta}\right)|z|^{n}\leq\frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}$$

$$\Rightarrow |z|^{n}\leq\frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}\left(\frac{p-\eta}{n-p+\eta}\right).$$

The last inequality leads us immediately to the disc $|z|<R_2$, where $R_2$ is given by (3.3).

**Theorem 3.3.** Let $f\in M^{p}(\alpha,\mu,\lambda,A,B)$. Then $f$ is $p$- valently convex of order $\eta$ $(0\leq\eta<p)$ in $|z|<R_3$, where

$$R_3=\inf_{n\in\mathbb{N}}\left\{\left[\frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}\left(\frac{p(p-\eta)}{(n+p)(n-p+\eta)}\right)\right]\right\}^{\frac{1}{n}} \tag{3.5}$$

Proof: Given $f \in M^{p}(\alpha,\mu,\lambda,A,B)$ and $f$ is convex of order $\eta,\ (0 \le \eta < p)$ we have

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| < p-\eta. \tag{3.6}$$

For the left hand side of (3.6), we have

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| = \left|\frac{\sum_{n=1}^{\infty} n(n+p)a_{p+n}z^{p+n-1}}{pz^{p-1}+\sum_{n=1}^{\infty}(n+p)a_{p+n}z^{p+n-1}}\right| \le \frac{\sum_{n=1}^{\infty} n(n+p)a_{p+n}z^{n}}{p+\sum_{n=1}^{\infty}(n+p)a_{p+n}z^{n}}.$$

The last expression is less than $(p-\eta)$ if

$$\frac{\sum_{n=1}^{\infty} n(n+p)a_{p+n}z^{n}}{p+\sum_{n=1}^{\infty}(n+p)a_{p+n}z^{n}} < p-\eta$$

which implies

$$\sum_{n=1}^{\infty}\left(\frac{(n+p)(n-p+\eta)}{p(p-\eta)}\right)a_{p+n}\,|z|^{n} < 1.$$

Using the fact that, $f \in M^{p}(\alpha,\mu,\lambda,A,B)$ if and only if

$$\sum_{n=1}^{\infty}\frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}a_{n+p} \le 1,$$

we can say that (3.4) is true if

$$\left(\frac{(n+p)(n-p+\eta)}{p(p-\eta)}\right)|z|^{n} \le \frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}$$

$$\Rightarrow |z|^{n} \le \frac{\left[1-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(1+p)\lambda\right](A+1)+B(1+2p)\right]}{p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]}\left(\frac{p(p-\eta)}{(n+p)(n-p+\eta)}\right).$$

The last inequality leads us immediately to the disc $|z| < R_3$, where $R_3$ is given by (3.5).

## 4. Weighted Mean, Arithmetic Mean and Linear Combination

Recently, W.G. Asthan, H.D. Mustafa and E.K. Mouajeeb [1] proved Weighted Mean, Arithmetic Mean and linear Combination of regular functions.

**Definition 4.1.** Let $P,Q \in D_p(A,B)$ then the weighted mean $W_{PQ}$ of $P$ and $Q$ is defined as

$$W_{PQ} = \frac{1}{2}\left[(1-m)P(z)+(1+m)Q(z)\right], \qquad (0<m<1). \tag{4.1}$$

**Definition 4.2.** Let $f_j(z) = z^{p} + \sum_{n=1}^{\infty} a_{n+p,\,j}z^{n+p},\ \ (j=1,2,3,\ldots,m)$ be the function in the class $M^{p}(\alpha,\mu,\lambda,A,B)$ then the arithmetic mean of $f_j,\ (j=1,2,3,\ldots,m)$ is defined by

$$g(z) = \frac{1}{m}\sum_{j=1}^{m} f_j(z). \tag{4.2}$$

**Definition 4.3.** Let $f_j(z) = z^{p} + \sum_{n=1}^{\infty} a_{n+p,\,j}z^{n+p},\ \ (j=1,2,3,\ldots,m)$ be the function in the class $M^{p}(\alpha,\mu,\lambda,A,B)$ then the linear combination of $f_j,\ (j=1,2,3,\ldots,m)$ is defined by

$$G(z) = \frac{1}{m}\sum_{j=1}^{m} n_j f_j(z) \tag{4.3}$$

where $\sum_{j=1}^{m} n_j = 1.$

**Theorem 4.1.** Let $P, Q \in \mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$. Then the weighted mean $W_{PQ}$ of $P$ and $Q$ is also in the class $\mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$.

Proof: By definition 4.1 we have

$$\begin{aligned} W_{PQ} &= \frac{1}{2}\left[(1-m)P(z)+(1+m)Q(z)\right], \qquad (0<m<1) \\ &= \frac{1}{2}\left[(1-m)\left(z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\right)+(1+m)\left(z^{p}+\sum_{n=1}^{\infty}b_{p+n}z^{p+n}\right)\right] \\ &= z^{p}+\sum_{n=1}^{\infty}\frac{1}{2}\left[(1-m)a_{p+n}+(1+m)b_{p+n}\right]z^{p+n}. \end{aligned}$$

Since, $Q \in \mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$ so by theorem 2.1 we have

$$\sum_{n=1}^{\infty}\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]a_{n+p} \le p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]$$

and

$$\sum_{n=1}^{\infty}\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]b_{n+p} \le p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]$$

$$\begin{aligned} &\sum_{n=1}^{\infty}\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]\frac{1}{2}\left[(1-m)a_{p+n}+(1+m)b_{p+n}\right] \\ &= \frac{1}{2}(1-m)\sum_{n=1}^{\infty}\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]a_{n+p} \\ &\quad +\frac{1}{2}(1+m)\sum_{n=1}^{\infty}\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]b_{n+p} \\ &\le \frac{1}{2}(1-m)p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]+\frac{1}{2}(1+m)p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right] \\ &= p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]. \end{aligned}$$

Therefore $W_{PQ} \in \mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$.

Hence the proof of theorem is completed.

**Theorem 4.2.** Let $h_j(z)=z^{p}+\sum_{n=1}^{\infty}a_{n+p,\,j}z^{n+p}$, $(j=1,2,3,\ldots,m)$ be the function in the class $\mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$ then the arithmetic mean of $h_j$, $(j=1,2,3,\ldots,m)$ is defined by $g(z)=\frac{1}{m}\sum_{j=1}^{m}h_j(z)$ is also in the class $\mathrm{M}^{p}(\alpha,\mu,\lambda,A,B)$.

Proof: Since $h_j(z)=z^{p}+\sum_{n=1}^{\infty}a_{n+p,\,j}z^{n+p}$, $(j=1,2,3,\ldots,m)$. Therefore

$$\begin{aligned} g(z) &= \frac{1}{m}\sum_{j=1}^{m}h_j(z) \\ &= \frac{1}{m}\sum_{j=1}^{m}\left(z^{p}+\sum_{n=1}^{\infty}a_{n+p,\,j}z^{n+p}\right) \\ &= z^{p}+\sum_{n=1}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m}a_{n+p,\,j}\right)z^{n+p}. \end{aligned}$$

We have $h_j(z)=z^{p}+\sum_{n=1}^{\infty}a_{n+p,\,j}z^{n+p}$, $(j=1,2,3,\ldots,m)$. So by theorem 2.1 we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]\left(\frac{1}{m}\sum_{j=1}^{m}a_{n+p,\,j}\right) \\ &= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=1}^{\infty}\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]a_{n+p,\,j}\right) \\ &\le \frac{1}{m}\sum_{j=1}^{m}\left(p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right]\right) \end{aligned}$$

$$= p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right].$$

Hence the proof of the theorem is completed.

**Theorem 4.3.** Let $h_j(z) = z^p + \sum_{n=1}^{\infty} a_{n+p,\,j} z^{n+p}$, $(j = 1,2,3,...,m)$ be the function in the class $\mathrm{M}^{\,p}(\alpha,\mu,\lambda,A,B)$ then the linear combination of $h_j$, $(j = 1,2,3,...,m)$ is defined by $G(z) = \sum_{j=1}^{m} n_j h_j(z)$ where $\sum_{j=1}^{m} n_j = 1$ is also in the class $\mathrm{M}^{\,p}(\alpha,\mu,\lambda,A,B)$.

Proof: Since $h_j(z) = z^p + \sum_{n=1}^{\infty} a_{n+p,\,j} z^{n+p}$, $(j = 1,2,3,...,m)$. Therefore

$$\begin{aligned} G(z) &= \sum_{j=1}^{m} n_j h_j(z) \\ &= \sum_{j=1}^{m} n_j \left( z^p + \sum_{n=1}^{\infty} a_{n+p,\,j} z^{n+p} \right) \\ &= z^p + \sum_{n=1}^{\infty} \left( \sum_{j=1}^{m} n_j a_{n+p,\,j} \right) z^{n+p}. \end{aligned}$$

We have $h_j(z) = z^p + \sum_{n=1}^{\infty} a_{n+p,\,j} z^{n+p}$, $(j = 1,2,3,...,m)$. So by theorem 2.1 we have

$$\sum_{n=1}^{\infty}\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right]\left(\sum_{j=1}^{m} n_j a_{n+p,\,j}\right)$$

$$= \sum_{j=1}^{m} n_j \left( \sum_{n=1}^{\infty}\left[n-\left[p\left((p-\alpha)\mu+\alpha\lambda\right)+(n+p)\lambda\right](A+1)+B(n+2p)\right] a_{n+p,\,j} \right)$$

$$\leq \sum_{j=1}^{m} n_j \left( p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right] \right)$$

$$= p(A+1)\left[(p-\alpha)\mu+\lambda(\alpha-1)\right].$$

where $\sum_{j=1}^{m} n_j = 1$, hence the proof of the theorem is completed.

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