International Journal of Multidisciplinary Education and Research
ISSN: 2455-4588; Impact Factor: RJIF 5.12
www.educationjournal.in
Volume 1; Issue 7; September 2016; Page No. 04-17

Some subclasses of meromorphic P-valent functions with positive coefficients defined on the punctured disc using a differential operator

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Abstract

Motivated by the works of M.K. Aouf and others, we in this paper introduce and study a generalized subclass of meromorphic p-valent functions using a differential operator. We obtain interesting results such as coefficient estimates, growth and distortion bounds and prove closure and integral transforms theorems and other related results.

Keywords: Analytic, meromorphic p-valent, subordination, radius of convexity

1. Introduction

Let $U^* = \{z \in \mathrm{C}: 0 < |z| < 1\}$. Let $f(z)$ be analytic and meromorphic p-valent defined on U^* of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k}\ z^{p+k} \qquad (a_{p+k} \geq 0; p \in \mathrm{N}). \tag{1.1}$$

We denote by $\sum_{p}^{*}$, the class of all functions analytic and meromorphic p-valent of the form (1.1). In [4] M.K. Aouf introduced the differential operator as follows: For a function $f(z) \in \sum_{p}^{*}$ 1 , define the differential operator $D_{\lambda,p}^{n}$ by

$$D_{\lambda,p}{}^{0} f(z) = f(z),$$

$$\begin{aligned} D_{\lambda,p}^{1} f(z) &= (1-\lambda) f(z) + \frac{\lambda}{p} z f'(z) + \frac{2\lambda}{z^p} \\ &= \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{p+\lambda k}{p}\right) a_{p+k}\ z^{p+k} \end{aligned}$$

$$= D_{\lambda,p} f(z) \ (\lambda \geq 0; p \in \mathrm{N}),$$

$$D_{\lambda,p}2\ f(z) = D_{\lambda,p}\ (D_{\lambda,p}1\ f(z))$$

and

$$D_{\lambda,p}^{n} f(z) = D_{\lambda,p}\ (D_{\lambda,p}^{n-1} f(z))$$

$$= (1-\lambda)\ (D_{\lambda,p}^{n-1} f(z)) + \frac{\lambda}{p} z\ (D_{\lambda,p}^{n-1} f(z))' + \frac{2\lambda}{z^p}$$

$$(\lambda \geq 0; n, p \in \mathrm{N}). \tag{1.2}$$

By a simple computation, it is easy to verify:

$$D_{\lambda,p}^{n} f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(\frac{p+\lambda k}{p}\right)^{n} a_{p+k}\ z^{p+k} \quad (n \in N_0 = \mathbb{N} \cup \{0\}; p \in \mathbb{N}) . \tag{1.3}$$

With $F(z) = \sum_{k=0}^{\infty} \left(\frac{p+\lambda k}{p}\right)^{n} z^{p+k}$, interms of Hadamard product, (1.3) can be put in the form,

$$D^n_{\lambda,p}f(z) = (f * F)(z) = \sum_{k=0}^{\infty} \left(\frac{p+\lambda k}{p}\right)^n \; a_{p+k} \;\; z^{p+k} \quad . \tag{1.4}$$

Making use of this operator Aouf defined a subclass of meromorphic multivalent functions with positive coefficients as in:

$$f(z) \in \sum_{p}^{*} \; n$$

Definition 1.1. *A function belongs to the class F* (*α, β, γ*) *if it λ ,p satisfies the following inequality:*

$$\left| \frac{z^{p+1} \;\; (D^n_{\lambda,p}f(z))' \; + \; p}{(2\gamma-1) \; z^{p+1} \;\; (D^n_{\lambda,p}f(z))' \; + \; (2\gamma\alpha - p)} \right| < \beta \;\; ,$$

$$(\alpha \; (0 \le \alpha < P) \; ; \beta \;\; (0 < \beta \;\; \le 1) \; ; \; \gamma(\tfrac{1}{2} \le \gamma \le 1); \; \lambda \ge 0 \; ; p \in \mathbb{N};$$

n ∈ N0). *Besides this, several authors have done extensive work on subclass of meromorphic multivalent functions. For a brief survey one can refer to* [6, 9, 10, 15, 16].
We in this paper, generalize Aouf's work using the principle of subordination as given below:

$$f(z) \in \sum_{p}^{*} \qquad *,n$$

Definition 1.2. *A function is said to belong to the class* $^{P}(A,B)$ *λ, p if it satisfies the subordination*

$$\frac{-1}{p} \; z^{p+1} \;\; (D^n_{\lambda,p}f(z))' \;\; \prec \;\; \frac{1+Az}{1+Bz} \qquad 0 < |z| < 1, \qquad -1 \le A < B \le 1.$$

We observe the following

For $n = 0 \;\; , \;\; \lambda = 0 \; , \;\; p = 1, \; A = \frac{(p-2\alpha\gamma)\beta}{p}, \;\; B = (2\gamma - 1)\beta \;\; ,$

(i) $\displaystyle\sum_{0,1}^{*,0}(A,B) \; = \; \sum_{0,1}^{*,0}\left(\frac{(p-2\alpha\gamma)\beta}{p}, (2\gamma-1)\beta \;\right) \; = \; \overset{0}{\underset{0,1}{F}}(\alpha\beta \;\; ,\gamma) \; = \; \sum_{d}(\alpha\beta \;\; ,\gamma)$
the class introduced and studied by Cho *et al.* [5].

(ii) $\displaystyle\sum_{\lambda,p}(A,B) \; = \; \sum_{\lambda,p}\left(\frac{(p-2\alpha\gamma)\beta}{p}, (2\gamma-1)\beta \;\right) = \overset{n}{\underset{\lambda,p}{F}}(\alpha\beta \;\; ,\gamma)$
$*,n$ $*,n$, the class introduced and studied by Aouf [4].

(iii) $\displaystyle\sum_{\lambda,p}^{*,n}\left(\frac{(p-2\alpha)}{p}, 1\right) \; = \; \sum_{\lambda,p}^{*,n}(\alpha) \; = \; \{f(z) \in \sum_{p}^{*} : Re\{-z^{p+1}(D^n_{\lambda,p}f(z))'\} \; >$
$\alpha, 0 \le \alpha < p, \; p \in \mathbb{N}, n \in N_0 z \in U^*\};$

(iv) $\displaystyle\sum_{0,p}^{*,0}\left(\frac{(p-2\alpha)}{p}, 1\right) \; = \; \sum_{p}^{*}(\alpha) = \{f(z) \in \sum_{p}^{*} : Re\{-z^{p+1}f'(z)\} \; > \; \alpha, \; 0 \le$
$\alpha < p, p \in \mathbb{N}, z \in U^*\};$

(v) $\displaystyle\sum_{0,p}^{*,0}(\alpha\beta \;\; ,1) \; = \; \sum_{p}^{*}(\alpha\beta \;\;) \; = \; \{f(z) \in \sum_{p}^{*} : \left|\frac{z^{p+1}f'(z)+p}{z^{p+1}f'(z)+2\alpha-p}\right| \; < \; 1; z \in$
$U^*, \; 0 \le \alpha < p, \; 0 < \beta \;\; \le 1, \; p \in \mathbb{N}, z \in U^*\}.$

In this paper coefficient problems, distortion bounds, radius of convexity, closure theorem and integral transforms are investigated for the $*,n$ class $^{P}(A,B)$. We also prove theorems involving convolution products.
λ,p

2. Main Results

$* \; *,n$

Theorem 2.1. *Let* $f(z) \in {}^{P}$. *Then* $f \in {}^{P}(A,B)$ *if and only if*

p λ,p

$$(B+1)\sum_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}\ a_{p+k}\ \le\ (A+B)p,$$

$$for 1\le\ A\ <\ B\le 1,\ \ \lambda\ge 0,\ n\in N_0,\ p\in\mathbb{N}.$$

(2.1)

$*,n$

Proof. Assume that $f \in {}^{\mathrm{P}}(A, B)$. Then λ,p

$$\left|\frac{z^{p+1}(D^{n}_{\lambda,p}(A,B)f(z))'+p}{Bz^{p+1}(D^{n}_{\lambda,p}(A,B)f(z))'+Ap}\right| = \left|\frac{\sum\limits_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}z^{p+k}}{B(p-\sum\limits_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}z^{p+k}+Ap)}\right| < 1$$

,

($z \in U^*$). Applying the relation $Re z \le |z|\ \forall\ z$, we have

$$Re\left\{\frac{\sum\limits_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}z^{2p+k}}{(A+B)p-\sum\limits_{k=0}^{\infty}B(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}z^{2p+k}}\right\}\ <\ 1\quad (z\in U^*).\ \ (2.2)$$

Now choose the values of z on the real axis so that $z^{p+1}(D_{\lambda,p}{}^{n} f(z))^{0}$ is real.
Letting $z \to 1^{-}$ through positive values, from (2.2) we obtain

$$(B+1)\sum_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}\le(A+B)p.$$

Conversely, assume that the inequality (2.1) holds.
This implies

$$\left|\sum_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}z^{2p+k}\right|$$
$$-\left|B(p-\sum_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}z^{2p+k})+Ap\right|\ <\ 0$$

.

(2.3)

For $|z| = r < 1$ the left hand side of (2.3) is bounded above by

$$\sum_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}r^{2p+k}$$
$$-B(p-\sum_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}r^{2p+k})-Ap<0$$

$$<(B+1)\sum_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}-(A+B)p\le 0$$

.

Thus

$$\left|\frac{z^{p+1}(D^{n}_{\lambda,p}f(z))'+p}{Bz^{p+1}(D^{n}_{\lambda,p}f(z))'+Ap}\right|<1$$

.

$*,n$

Hence $f(z) \in {}^{\mathrm{P}}(A,B)$. λ,p

$*$ $*,n$

Corollary 2.2. *The function* $f \in {}^{\mathrm{P}}$ *is in* ${}^{\mathrm{P}}(A,B)$ *if*

p λ,p

$$a_{p+k} \le \frac{(A+B)P}{(B+1)(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}},$$

$$(-1 \le A < B \le 1, \lambda \ge 0,\ p \in \mathbb{N},\ n \in N_0).$$

This result is sharp for the function f(z) given by

$$f(z) = \frac{1}{z^p} + \frac{(A+B)p}{(B+1)(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}} z^{p+k},$$

$$(-1 \le A < B \le 1, \lambda \ge 0,\ p \in \mathbb{N},\ n \in N_0).$$

Letting $n = \lambda = 0$ *in theorem we get,*

$$\sum_{\lambda,p}^{*,n}\left(\frac{p-2\alpha}{p}, 1\right) \quad (0 \le \alpha < p)$$

$*, 0$

Corollary 2.3. *The function* $f \in {}^{\mathrm{P}}(\alpha,1,1)$ *is in* $0,p$ *if and only if*

$$\sum_{k=0}^{\infty}(p+k)a_{p+k} \le \frac{p-2\alpha+1}{2}.$$

Note that when $p = 1$,

∞

X

$(k + 1)\, a_{k+1} \le (1 - \alpha)$

$k=0$ *the conditions is meromorphic univalent function.*

Corollary 2.4. *Upon specialising* $A = \frac{(p-2\gamma\alpha)\beta}{p}$, $B = (2\gamma-1)\beta$ *this reduces to*

$$\sum_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(1+2\beta\gamma-\beta)a_{p+k} \le 2\beta\gamma(p-\alpha)$$

the result obtained in Aouf [4].

Theorem 2.5. *(Distortion theorem)*

$* *, n$

The function $f \in {}^{\mathrm{P}}$ *is in* ${}^{\mathrm{P}}(A,B)$ *then for* $0 < |z| = r < 1$

p λ,p

$$\frac{1}{r^p} - \frac{(A+B)}{(B+1)}r^p \le |f(z)| \le \frac{1}{r^p} + \frac{(A+B)}{(B+1)}r^p,$$

$$\frac{p}{r^{p+1}} - \frac{(A+B)}{(B+1)}r^{p-1} \le |f'(z)| \le \frac{p}{r^{p+1}} + \frac{(A+B)}{(B+1)}r^{p-1}$$

(2.4) (2.5)

The bounds in (2.4) and (2.5) are in

$$f(z) = \frac{1}{z^p} + \frac{(A+B)}{(B+1)}z^p \quad (p \in \mathbb{N}). \tag{2.6}$$

Proof. By the theorem 1, we have

$$p(B+1)\sum_{k=0}^{\infty}a_{p+k} \le \sum_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)a_{p+k} \le (A+B)p,$$

that is

$$\sum_{k=0}^{\infty} a_{p+k} \leq \frac{A+B}{B+1}$$

$$< \ |z| \ = \ r \ < \ 1, \qquad . \qquad (2.7)$$

Thus 0

$$|f(z)| \ \leq \ \frac{1}{r^p} + \sum_{k=0}^{\infty} a_{p+k} \ r^{p+k} \ \leq \ \frac{1}{r^p} + r^p \ \sum_{k=0}^{\infty} a_{p+k} \ \leq \ \frac{1}{r^p} + \frac{A+B}{B+1} r^p \qquad (2.8)$$

, and

$$|f(z)| \ \geq \ \frac{1}{r^p} - \sum_{k=0}^{\infty} a_{p+k} \ r^{p+k} \ \geq \ \frac{1}{r^p} - r^p \ \sum_{k=0}^{\infty} a_{p+k} \ \geq \ \frac{1}{r^p} - \frac{A+B}{B+1} r^p. \qquad (2.9)$$

The inequalities (2.8) and (2.9) together yield (2.4) and it is follows theorem 1 that

$$\sum_{k=0}^{\infty} (p+k) \ a_{p+k} \ \leq \ \frac{(A+B)p}{B+1} \ . \qquad (2.10)$$

Hence

$$|f'(z)| \ \leq \ \frac{p}{r^p+1} \ + \ \sum_{k=0}^{\infty} (p+k) \ a_{p+k} \ r^{p+k-1}$$

$$\leq \ \frac{p}{r^{p+1}} \ + \ r^{p-1} \ \sum_{k=0}^{\infty} (p+k) \ a_{p+k} \ \leq \ \frac{p}{r^{p+1}} \ + \ \frac{(A+B)p}{B+1} \ r^{p-1}, \qquad (2.11)$$

and

$$|f'(z)| \ \geq \ \frac{p}{r^p+1} \ - \ \sum_{k=0}^{\infty} (p+k) \ a_{p+k} \ r^{p+k-1}$$

$$\geq \ \frac{p}{r^{p+1}} \ - \ r^{p-1} \ \sum_{k=0}^{\infty} (p+k) \ a_{p+k} \ \geq \ \frac{p}{r^{p+1}} \ - \ \frac{(A+B)p}{B+1} \ r^{p-1}. \qquad (2.12)$$

The inequalities (2.11) and (2.12) together yield (2.5).It clear seen that the function *f*(*z*) defined by (2.6) is extremal for theorem 2.

Corollary 2.6. *Upon specialising* $A = \frac{(p-2\gamma\alpha)\beta}{p}$, $B = (2\gamma - 1)\beta$ *this reduces to*

$$\frac{1}{r^p} - \frac{2\beta \ \gamma(p-\alpha)}{p(1+2\beta \ \gamma - \beta \ )} r^p \ \leq \ |f(z)| \ \leq \ \frac{1}{r^p} \ + \ \frac{2\beta \ \gamma(p-\alpha)}{p(1+2\beta \ \gamma - \beta \ )} r^p,$$

$$\frac{p}{r^{p+1}} \ - \ \frac{2\beta \ \gamma(p-\alpha)}{(1+2\beta \ \gamma - \beta \ )} r^{p-1} \ \leq \ |f'(z)| \ \leq \ \frac{p}{r^{p+1}} \ + \ \frac{2\beta \ \gamma(p-\alpha)}{(1+2\beta \ \gamma - \beta \ )} r^{p-1},$$

And

$$f(z) \ = \ \frac{1}{z^p} \ + \ \frac{2\beta \ \gamma(p \ - \ \alpha)}{p(1 \ + \ 2\beta \ \gamma \ - \beta \ )} z^p \quad (p \in \mathbb{N})$$

the result obtained in Aouf [4].

Theorem 2.7. *(Radius of convexity)*
∗ ∗, *n*
Assume that the function f ∈ $^{\mathrm{P}}$ *is in* $^{\mathrm{P}}$ *(A, B), then f(z) is*
P λ, p meromorphically multivalent convex of order ρ (0 ≤ *ρ* ≤ *p*) *in*
0 < |*z*| < *r* (*λ, p, n, A, B*) *where*

$$r(\lambda, \ p, \ n, \ A, \ B) \ = \ \inf_{k} \left[\frac{(p \ - \ \rho) \left(\frac{p+\lambda k}{p}\right)^n (B+1)}{(A+B)(3p+k-\rho)} \right]^{\frac{1}{2p+k}} ,$$

$(k \geq 0;\ p \in \mathbb{N};\ n \in N_0)$. (2.13)

the result is sharp. Proof. It is enough to prove that

$$\left|\frac{(zf'(z))' + pf'(z)}{f'(z)}\right| \leq p-\rho \quad for\ 0 < |z|\ r(\lambda,\ \rho,\ n,\ A,\ B,\ p).$$

$$\left|\frac{(zf'(z))' + pf'(z)}{f'(z)}\right| = \left|\frac{\sum\limits_{k=0}^{\infty}(p+k)^2\ a_{p+k}z^{p+k-1} + \sum\limits_{k=0}^{\infty}(p+k)p\ a_{p+k}z^{p+k-1}}{\frac{-p}{z^{p+1}} + \sum\limits_{k=0}^{\infty}(p+k)\ a_{p+k}z^{p+k-1}}\right|$$

$$\leq \frac{\sum\limits_{k=0}^{\infty}a_{p+k}\ (p+k)\ (2p+k)\ r^{2p+k}}{p - \sum\limits_{k=0}^{\infty}(p+k)\ a_{p+k}\ r^{2p+k}}. \quad |z| < r = 1$$

.

$$\Rightarrow \sum_{k=0}^{\infty}a_{p+k}\ (p+k)\ (2p+k)\ r^{2p+k} \leq (p-\delta)(p - \sum_{k=0}^{\infty}(p+k)\ a_{p+k}\ r^{2p+k}).$$

$$\frac{\sum\limits_{k=0}^{\infty}a_{p+k}\ (p+k)\ (3p+k-\rho)}{p(p - \rho)} \leq 1. \qquad (2.14)$$

By theorem (1)

$$\frac{(B+1)\sum\limits_{k=0}^{\infty}(p+k)\ \left(\frac{p+\lambda k}{p}\right)^n\ a_{p+k}}{(A+B)p} \leq 1 \qquad (2.15)$$

.

$$\frac{(3p+k-\rho)r^{2p+k}}{(p - \rho)} \leq \frac{(B+1)\ \left(\frac{p+\lambda k}{p}\right)^n}{(A+B)}.$$

$$r \leq \left[\frac{\left(\frac{p+\lambda k}{p}\right)^n (B+1)(P-\rho)}{(A+B)(3p+k-\rho)}\right]^{\frac{1}{2p+k}} \quad (k \geq 0;\ p \in \mathbb{N};\ n \in \mathbb{N}_0). \quad (2.16)$$

The result (4.1) is sharp with extremal function *f*(*z*) given by (2.5).

Corollary 2.8. *Upon specialising* $A = \frac{(p-2\gamma\alpha\beta)}{p}$, $B = (2\gamma-1)\beta$ *this reduces to*

$$r(\lambda,\ p,\ n,\ \alpha\beta\ ,\ \gamma) = \inf_k \left[\frac{p\ (p - \alpha)\left(\frac{p+\lambda k}{p}\right)^n (1+2\beta\ \gamma-\beta\ )}{2\beta\ \gamma(p-\alpha)(3p+k-\delta)}\right]^{\frac{1}{2p+k}}$$

, $(k \geq 0;\ p \in \mathbb{N};\ n \in \mathbb{N}0)$. *the result obtained in Aouf* [4].

Theorem 2.9. *(Closure theorems)*

Let $f_{p-1}(z) = \frac{1}{z^p}$ (2.17)

And

$$f_{p+k}(z) = \frac{1}{z^p} + \frac{(A+B)\ P}{(p + k)\left(\frac{p+\lambda k}{p}\right)^n (B+1)}\ z^{p+k} \quad (k \geq 0;\ p \in \mathbb{N};\ n \in \mathbb{N}_0)$$

$$(2.18)$$

$*,n$

.

Then f(z) *be in the class* $^{\mathrm{P}}$(*A,B*) *if and only if*
λ,p

∞ *f*(z) = $^{\mathrm{X}}$ $\Theta_{p+k}f_{p+k}$(z), (2.19)

k=−1

Where

$$\Theta_{p+k} \;\geq\; 0 \quad and \quad \sum_{k=-1}^{\infty}\Theta_{p+k} \;=\; 1 \quad .$$

∞ ∞

Proof. let $f(z) = {}^{\mathrm{P}}\,\Theta_{p+k}f_{p+k}(z)$ where$\Theta_{p+k} \geq 0$ and ${}^{\mathrm{P}}\,\Theta_{p+k} = 1$. k=−1 k=−1

Then

$$f(z) = \sum_{k=-1}^{\infty}\Theta_{p+k}f_{p+k}(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty}\Theta_{p+k}\frac{(A+B)p}{(p+k)\left(\frac{p+\lambda k}{p}\right)^n(B+1)}z^{p+k} \quad .$$

Then

$$\sum_{k=0}^{\infty}\Theta_{p+k}\frac{(A+B)p}{(p+k)\left(\frac{p+\lambda k}{p}\right)^n(B+1)} \;\cdot\; \frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^n(B+1)}{(A+B)p}$$

$$= \;\sum\Theta_{p+k} \;=\; 1-\Theta_{p-1} \;\leq\; 1,$$

$*,n$ this imply that $f(z) \in {}^{\mathrm{P}}(A,B)$.

λ,p

Conversely

$*,n$

suppose $f(z) \in {}^{\mathrm{P}}(A,B)$ then

λ,p

$$a_{p+k} \;\leq\; \frac{(A+B)p}{(p+k)\left(\frac{p+\lambda k}{p}\right)^n(B+1)} \quad (k \geq\; 0;\; p \in \mathbb{N};\; n \in \mathbb{N}_0) \quad .$$

Let

$$\Theta_{p+k} \;\leq\; \frac{(A+B)p}{(p+k)\left(\frac{p+\lambda k}{p}\right)^n(B+1)}\; a_{p+k} \quad (k \geq\; 0;\; p \in \mathbb{N};\; n \in \mathbb{N}_0)$$

and

$$\Theta_{p-1} \;=\; 1 - \;\sum_{k=0}^{\infty}\Theta_{p+k} \quad ,$$

∞

this imply that $f(z) = {}^{\mathrm{P}}\Theta_{p+k}f_{p+k}(z)$.

k=0

Corollary 2.10. *Upon specialising* $A \;=\; \frac{(p-2\gamma\alpha)\beta}{p}\;,\; B \;=\; (2\gamma-1)\beta$ *this reduces to*

$$f(z) = \sum_{k=0}^{\infty}\mu_{p+k}\;\; f_{p+k}(z), \quad where\mu_{p+k} \geq 0 \;\; and \;\; \sum_{k=-1}^{\infty}\mu_{p+k} = 1$$

the result obtained in Aouf [4].

Theorem 2.11. *Let*

$$f_1(z) = \frac{1}{z^p} \;+\; \sum_{k=0}^{\infty}a_{p+k,1}\;\; z^{p+k} \quad (a_{p+k,1} \geq\; 0) \quad ,$$

$$f_2(z) = \frac{1}{z^p} \;+\; \sum_{k=0}^{\infty}a_{p+k,2}\;\; z^{p+k} \quad (a_{p+k,2} \geq\; 0) \;\in\; \sum_{\lambda,p}^{*,n}(A,B) \quad .$$

Then $f(z) = (1-\upsilon)f_1(z) + \upsilon f_2(z)$ $(0 \leq \upsilon \leq 1)$ *is also in.*

Proof. Assume that

$$f_j(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,j} \; z^{p+k} \quad (a_{p+k,j} \geq 0;\ j = 1, 2)$$

$*$,*n* is in the class^{P}(*A*,*B*). It is enough to prove that
λ,*p* $*$,*n*

q(*z*) = (1 − *υ*)*f*$_1$(*z*) + *υf*$_2$(*z*) (0 ≤ *υ* ≤ 1) and *q*(*z*) ∈ ^{P}(*A*,*B*).
λ,*p*

$$q(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} [(1-\upsilon)a_{p+k,1} + \upsilon a_{p+k,2}] \; z^{p+k} \quad (0 \leq \upsilon \leq 1)$$

,

By the theorem 1, we have

$$\sum_{k=0}^{\infty} (p+k) \left(\frac{p+\lambda k}{p} \right)^n (B+1) [(1-\upsilon)a_{p+k,1} + \upsilon a_{p+k,2}]$$

$$\leq (1-\upsilon)(A+B)p + \upsilon(A+B)p,$$

$$= (A+B)p,$$

$*,n$ this imply that *q*(*z*) ∈ ^{P}(*A*,*B*).
λ,*p*

$$A = \frac{(p-2\gamma\alpha)\beta}{p} \ , \ B = (2\gamma-1)\beta$$

Corollary 2.12. *Upon specialising* $*,n$ *this reduces to h*(z) = (1−*t*)*f*$_1$(z) + *tf*$_2$(z) 0 ≤ *t* ≤ 1 ∈ ^{P}(*A*,*B*) *the result λ,p obtained in Aouf* [4].

Theorem 2.13. *(integral transforms)*
If

$$h_{c+p-1}(z) = c \int_0^1 \upsilon^{c+p-1} f(\upsilon z) d\upsilon \quad 0 < c < \infty$$

0 (2.20)

is in the class F (*μ*,1,1), 0 ≤ *μ* < *p where*
0,*p*

$$\mu = \mu(p, A, B, c) = \frac{p\,[(2p+c)(B+1) - (A+B)c]}{(2p+c)(B+1)} \tag{2.21}$$

Proof.

$$Let \quad f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \in \sum_{\lambda,p}^{*,n} (A, B)$$

.

Then

$$h_{c+p-1}(z) = c \int_0^1 \upsilon^{c+p-1} f(\upsilon z) d\upsilon.$$

$$c \int_0^1 \upsilon^{c+p-1} f(\upsilon z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \frac{c}{2p+k+c}$$

. $*$ *h*$_{c+p-1}$(*z*) are in class ^{P}(*μ*) 0 ≤ *μ* < *p* by corollary-2

p

$$\sum_{k=0}^{\infty} \frac{(k+p)c}{2p+k+c} a_{p+k} \leq p - \mu.$$

$$\sum_{k=0}^{\infty} \frac{(k+p)c}{(p-\mu)(2p+k+c)} a_{p+k} \leq 1. \tag{2.22}$$

$$\frac{(B+1)\sum\limits_{k=0}^{\infty}(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}a_{p+k}}{(A+B)p} \leq 1 \qquad (2.23)$$

$$\mu - p \leq \frac{-(A+B)p(k+p)c}{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)(2p+k+c)}.$$

$$\mu \leq \frac{p(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)(2p+k+c)-(A+B)p(k+p)c}{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)(2p+k+c)}. \qquad (2.24)$$

Put $k = 0$ in (2.24) we get

$$\mu \leq \frac{p(2p+c)(B+1)-(A+B)pc}{(2p+c)(B+1)}.$$

Corollary 2.14. *Upon specialising* $A = \frac{(p-2\gamma\alpha\beta}{p}$, $B = (2\gamma - 1)\beta$ *this reduces to*

$$\theta = \theta(p,\alpha\beta,\gamma,c) = \frac{p(2p+c)(1+2\beta\gamma-\beta)-2\beta\gamma(p-\alpha)c}{(2p+c)(1+2\beta\gamma-\beta)}$$

the result obtained in Aouf [4].

Theorem 2.15. *(Convolution properties)*

$$f_1(z) = \frac{1}{z^p} + \sum_{k} a_{p+k,1}z^{p+k} \quad (a_{p+k,1} \geq 0)$$

∞

Ifand

=0

$$f_2(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,2}z^{p+k}$$

$*,n$ $*,n$

$(a_{p+k,1} \geq 0)$ *are in* $^{P}(A,B)$ *then* λ,p

$(f_1 * f_2)(z)$ *is also in* $^{P}(\psi,A,B)$*. The result is sharp for the functions*

λ,p

$f_j(z)$ (j = 1,2) *given by*

$$f_j(z) = \frac{1}{z^p} + \frac{(B+A)}{(B+1)}z^p \quad (j = 1,2;\ p \in \mathbb{N}) \qquad (2.25)$$

Proof. By schild and silverman [12]. It is enough to prove theorem 7,we can find the ψ such that

$$\frac{\sum\limits_{k=0}^{\infty}(k+p)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)a_{p+k,1}a_{p+k,2}}{2\left(\frac{Ap+\alpha B}{p-\alpha}\right)\left(\frac{1}{2}+\frac{B(p-\alpha)}{2(Ap+\alpha B)}\right)(p-\psi)} \leq 1$$

$$\frac{\sum\limits_{k=0}^{\infty}(k+p)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)a_{p+k,1}a_{p+k,2}(p-\alpha)}{(A+B)p(p-\psi)} \leq 1$$

$$for\ \ f_j(z) \in \sum_{k=0}^{\infty}(\psi,A,B)(j=1,2). \qquad (2.26)$$

$*,n$

we know that $f_j(z) \in {}^{P}(A,B)$ (j = 1,2)

λ,p

$$\frac{\sum\limits_{k=0}^{\infty}(k+p)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}{(A+B)p}a_{p+k,j} \leq 1(j=1,2). \qquad (2.27)$$

By the cauchy schwarz inequality, we obtain

$$\left| \frac{\sum\limits_{k=0}^{\infty}(k+p)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}{(A+B)p} a_{p+k,1} \;+\; \frac{\sum\limits_{k=0}^{\infty}(k+p)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)^{2}}{(A+B)p} a_{p+k,2} \right|$$

$$\leq \frac{(k+p)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}{(A+B)p}\sqrt{a_{p+k,1}a_{p+k,2}} \;\leq\; 1.$$

$$\frac{a_{p+k,1}a_{p+k,2}(p-\alpha)}{p-\psi} \;\leq\; \sqrt{a_{p+k,1}a_{p+k,2}} \quad (k \geq 0\,;p\in\mathbb{N})$$

(2.28) (2.29)

(or)equivalently

$$\sqrt{a_{p+k,1}a_{p+k,2}} \;\leq\; \frac{p-\psi}{p-\alpha} \quad (k\geq 0).$$

(2.30)

By the inequality (2.28) it is enough to prove that

$$\sqrt{a_{p+k,1}a_{p+k,2}} \;\leq\; \frac{(A+B)p}{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}.$$

$$\psi \;\leq\; p-\frac{(A+B)p(p-\alpha)}{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)} \quad (k\geq 0).$$

Defining the function $\zeta(k) = p-\frac{(A+B)p(p-\alpha)}{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)} \quad (k\geq 0)$. Now check that $\zeta(k)$ is an increasing function of k.

$$\psi \;\leq\; \zeta(0) = p-\frac{(A+B)(P-\alpha)}{B+1}$$

Corollary 2.16. *Upon specialising* $A = \frac{(p-2\gamma\alpha)\beta}{p}\;,\; B = (2\gamma-1)\beta$ *this reduces to*

$$f_j(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)}z^p \quad (j=1,2;p\in\mathbb{N})$$ *the result obtained in Aouf* [4].

Theorem 2.17. *Let the function* $f_1(z) = \frac{1}{z^p} + \sum\limits_{k=0}^{\infty} a_{p+k,1}\, z^{p+k}$

($a_{p+k,1}\geq 0$*) and* $f_2(z) = \frac{1}{z^p} + \sum\limits_{k=0}^{\infty} a_{p+k,2}\, z^{p+k} \quad (a_{p+k,2}\geq 0) \;\in\; \sum\limits_{\lambda,p}^{*,n}(A,B)$.

$*,n$

Then $(f_1 * f_2)(z)$ *in* $^{P}(\Psi,A,B)$*where*

λ,p

$$\Psi \;=\; p-\frac{2(A_1+B)(A_2+B)}{B+1}.$$

The result is sharp in the function $f_j(z)$ $(j=1,2)$ *given by*

$$f_1(z) = \frac{1}{z^p} + \frac{(A_1+B)}{B+1} \quad (P \in \mathbb{N}) .$$

$$f_2(z) = \frac{1}{z^p} + \frac{(A_2+B)}{B+1} \quad (P \in \mathbb{N}) .$$

Theorem 2.18. *If*

$$f_1(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,1} z^{p+k} \ in \sum_{k=0}^{*,n} (A,B)$$

and

$$f_2(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,2} z^{p+k}$$

∗,n
with $|a_{p+k,2}| \leq 1$; $k = 0,1,2,...,p \in$ N, *then*$(f_1 * f_2)(z) \in {}^{\mathrm{P}}(A,B)$.
λ,p
Proof.

$$\sum_{k=0}^{\infty} \frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^n (B+1)}{(A+B)p} |a_{p+k,1} a_{p+k,2}| = \sum_{k=0}^{\infty} \frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^n (B+1)}{(A+B)p} a_{p+k,1} \leq 1 .$$

Theorem 1 imply that
∗,n
$(f_1 * f_2)(z) \in {}^{\mathrm{X}}(A,B)$.
λ,p

Corollary 2.19. *If*

$$f_1(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,1} z^{p+k} \ in \sum_{k=0}^{*,n} (A,B)$$

and

$$f_2(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,2} z^{p+k} \quad (0 \leq a_{p+k,2} \leq 1; \ k = 0,1,2,...,p \in \mathbb{N})$$

∗,n
then$(f_1 * f_2)(z) \in {}^{\mathrm{P}}(A,B)$.
λ,p

Theorem 2.20. *Let the functions*

$$f_1(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,1} z^{p+k}$$

and

$$f_2(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,2}$$

∗,n is in the class ${}^{\mathrm{P}}(A,B)$ *and*
k=0
$p(B+1) - 2(A+B)p \geq 0$. (2.31)
Then the function $q(z)$ *defined by*

$$q(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} (a^2_{p+k,1} + a^2_{p+k,2}) z^{p+k} \qquad (2.32)$$

∗,n
be in the class ${}^{\mathrm{P}}(A,B)$.
λ,p
∗,n

Proof. Given that $f_1(z) \in {}^{P}(A,B)$ we have
λ,p

$$\sum_{k=0}^{\infty} \frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}{(A+B)p} a_{p+k,1} \ \leq\ 1$$

and so

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}{(A+B)p}\right]^{2} a_{p+k,1}^{2} \ \leq\ 1$$

$*,n$.

similarly,$f(z) \in {}^{P}(A,B)$ we have
λ,p

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}{(A+B)p}\right]^{2} a_{p+k,2}^{2} \ \leq\ 1 .$$

Hence

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}{(A+B)p}\right]^{2} (a_{p+k,2}^{2} \ +\ a_{p+k,1}^{2}) \ \leq\ 1 .$$

By theorem 1, it is show that

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}{(A+B)p}\right] (a_{p+k,2}^{2} \ +\ a_{p+k,1}^{2}) \ \leq\ 1 . \qquad (2.33)$$

Hence the inequality (2.33) will be satisfied if, for k=1, 2,...

$$\frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}{(A+B)p} \ \leq\ \frac{1}{2}\left[\frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1)}{(A+B)p}\right]^{2} \qquad (2.34)$$

(or) if

$$(p+k)\left(\frac{p+\lambda k}{p}\right)^{n}(B+1) - 2(A+B)p \geq\ 0, \quad for\ \ k = 1, 2, ... \qquad (2.35)$$

the inequality (2.35) is a increasing function of k, thus the inequality (2.35) is satisfied for all k, if $p(B+1) - 2(A+B)p \geq 0$.

Corollary 2.21. *Upon specialising* $A\ =\ \frac{(p-2\gamma\alpha)\beta}{p}\ ,\ B\ =\ (2\gamma-1)\beta$ *this reduces to*

$$h(z) = \frac{1}{z^p}\ +\ \sum_{k=0}^{\infty}(a_{p+k,1}^{2}\ +\ a_{p+k,2}^{2})z^{p+k}$$

the result obtained in Aouf [4]. **Theorem 2.22.** *Let the functions*

$$f_1(z) = \frac{1}{z^p}\ +\ \sum_{k=0}^{\infty} a_{p+k,1} z^{p+k}$$

and

$$f_2(z) = \frac{1}{z^p}\ +\ \sum_{k=0}^{\infty} a_{p+k,2}$$

$*,n$ *is in the class*${}^{P}(A,B)$. *Then the function q(z)defined as (2.32) be in the*

λ,p
$*,n$
class $^{P}(\eta,A,B)$ where
λ,k

$$\eta = p - \frac{2(A+B)(p-\alpha)}{(B+1)} . \qquad (2.36)$$

This result is sharp for the functions $f_j(z) = \frac{1}{z^p} \ + \ + \frac{(B+A)}{(B+1)} z^p$ (j = 1, 2; $p \in$ N).
Proof.

$$\sum_{k=0}^{\infty} \frac{\left[(p+k)\left(\frac{p+\lambda k}{p}\right)^n (B+1)\right]^2}{[(A+B)p]^2} a_{p+k,j}^2 \ \leq \ \left[\sum_{k=0}^{\infty} \frac{(p+k)\left(\frac{p+\lambda k}{p}\right)^n (B+1)}{(A+B)p} a_{p+k,j}\right]^2 \ \leq \ 1$$

$$(j = 1, 2). \qquad (2.37)$$

The function $f_j(z) \ \ in \sum_{\lambda,p}^{*,n}(A,B) \quad (j=1,2)$, we have

$$\sum_{k=0}^{\infty} \frac{\left[(p+k)\left(\frac{p+\lambda k}{p}\right)^n (B+1)\right]^2}{2(B+1)^2} (a_{p+k,1}^2 \ + \ a_{p+k,2}^2) \ \leq \ 1 .$$

Thus, to find the η suchthat

$$\eta \ \leq \ p - \frac{2(A+B)p(p-\alpha)}{(p+k)\left(\frac{p+\lambda k}{p}\right)^n (B+1)} \quad (k \geq \ 0) . \qquad (2.38)$$

To defining a function $F(k) \ = \ p - \frac{2(A+B)p(p-\alpha)}{(p+k)\left(\frac{p+\lambda k}{p}\right)^n (B+1)} \quad (k \geq \ 0)$.

Thus $F(k)$ is an increasing function of k

$$\eta \ \leq \ F(0) = p - \frac{2(A+B)P(P-\alpha)}{P(B+1)} .$$

Corollary 2.23. *Upon specialising* $A \ = \ \frac{(p-2\gamma\alpha)\beta}{p} \ , \ B \ = \ (2\gamma - 1)\beta$ *this reduces to* $\Psi = p - \frac{4\beta\gamma(p-\alpha)^2}{p(1+2\beta\gamma-\beta)}$ *the result obtained in Aouf* [4].

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